Singularity confinement for maps with the Laurent property

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Abstract

The singularity confinement test is very useful for isolating integrable cases of discrete-time dynamical systems, but it does not provide a sufficient criterion for integrability. Quite recently a new property of the bilinear equations appearing in discrete soliton theory has been noticed: the iterates of such equations are Laurent polynomials in the initial data. A large class of non-integrable mappings of the plane are presented which both possess this Laurent property and have confined singularities. **MSC2000 numbers: 11B37, 93C10, 93C55**

There continues to be a great deal of interest in discrete-time dynamical systems that are integrable. There is a vast range of such systems, including symplectic maps and Bäcklund transformations for Hamiltonian systems in classical mechanics [1], mappings that preserve plane curves [2] which occur in statistical mechanics, discrete analogues of Painlevé transcendents [3], partial difference soliton equations appearing in numerical analysis and solvable quantum models [4], and equations arising in theories of discrete geometry and discrete analytic functions [5]. For some

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time it has been appreciated that integrability in the discrete setting is associated with certain weak growth phenomena [6], which can be measured by means of suitable notions of entropy [7] or complexity [8].

Given the multitude of application areas in which discrete integrable systems appear, the problem of identifying when a given system is integrable is of considerable importance. In the continuous setting, the Painlevé property has proved to be an extremely useful criterion for isolating integrable differential equations [9], and this led Grammaticos, Ramani and Papageorgiou to introduce the singularity confinement test for discrete equations [10]. However, while that test has been enormously successful at identifying discrete Painlevé equations, it turns out that singularity confinement is not sufficient for integrability, as was pointed out by Hietarinta and Viallet [7]. Those authors found numerous examples of rational maps, taking the form

$$x_{n+1} + x_{n-1} = f(x_n), \qquad (1)$$

which have confined singularities and yet whose orbit structure displays the characteristics of chaos. This led them to suggest that the stronger requirement of zero algebraic entropy (defined in terms of the growth of degrees of iterates) should be a necessary property of rational maps that are integrable, in agreement with the observations of Veselov [6]. More recently it was proposed by Ablowitz, Halburd and Herbst that the Painlevé property can be extended to difference equations using Nevanlinna theory [11], while Roberts and Vivaldi have used the orbit structure of rational maps defined over finite fields to detect integrability [12], and Halburd has translated the concepts of [11] into a Diophantine integrability criterion for discrete equations [13].

There is a large amount of literature on discrete bilinear equations, including the bilinear forms of discrete Painlevé equations [3], and bilinear partial difference equa-

tions such as Hirota's difference equation [4]. However, there is one aspect of such integrable bilinear equations that researchers on integrable systems have apparently overlooked, namely the fact that they have the *Laurent property*: that is, for suitably specified initial data, all of the iterates of these discrete equations are Laurent polynomials in these data with integer coefficients. It seems that this Laurent phenomenon was originally known only to a few people working in algebraic combinatorics, and for Hirota's equation it was first proved by Fomin and Zelevinsky within the framework of their theory of cluster algebras [14], with further combinatorial interpretations being found later [15].

One of the simplest examples of a bilinear equation displaying the Laurent phenomenon was found by Michael Somos, who considered kth order recurrences of the form

$$\tau_{n+k}\tau_n = \sum_{j=1}^{[k/2]} \tau_{n+k-j}\tau_{n+j}, \qquad k \geq 4, \tag{2}$$

taking the initial values $\tau_0 = \tau_1 = \ldots = \tau_{k-1} = 1$. Clearly each new iterate τ_{n+k} of (2) is a rational function of the initial data, so one expects $\tau_n \in \mathbb{Q}$, but it was observed numerically that for the Somos-4 recurrence

$$\tau_{n+4}\tau_n = \alpha\,\tau_{n+3}\tau_{n+1} + \beta\,(\tau_{n+2})^2 \tag{3}$$

with parameters $\alpha = \beta = 1$ and starting with four ones, an integer sequence

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \ldots$$

results. Several simple proofs of the integrality of this sequence were subsequently obtained (see the article by Gale and the other references [21]). However, it was realized that the deeper reason behind this lay in the fact that the recurrence (3) has the Laurent property, meaning that the iterates are polynomials in the coefficients,

the four initial data, and their inverses, and these Laurent polynomials have integer coefficients, i.e. $\tau_n \in \mathbb{Z}[\alpha, \beta, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}]$ for all n.

Fomin and Zelevinsky found that their theory of cluster algebras provided a suitable setting within which they could prove the Laurent property for a variety of discrete equations [14], including certain recurrences of the form

$$\tau_{n+k}\tau_n = F(\tau_{n+1}, \ldots, \tau_{n+k-1})$$

for particular choices of polynomials F (including Somos-k for $k = 4, 5, 6, 7$), as well as integrable two- and three-dimensional bilinear recurrences like Hirota's equation. As a parallel development, the connection between the iterates of the general Somos-4 recurrence (3) and sequences of points on elliptic curves has been explained by several people: two different approaches are found respectively in the PhD thesis of Swart and in the work of van der Poorten [16], while independently the author found the explicit solution of the initial value problem for both Somos-4 and Somos-5 [17] in terms of elliptic sigma functions. For earlier unpublished results of Zagier and Elkies on the associated elliptic curve and theta function formulae for the original Somos-5 sequence, see [18] and [19] respectively.

An essential observation in the work [17] was that both the fourth and the fifth order Somos recurrences could be understood in terms of a suitable integrable mapping of the plane. For example, setting $x_n = \tau_{n+1}\tau_{n-1}/(\tau_n)^2$ transforms (3) into the rational map

$$x_{n+1} x_{n-1} = \frac{\alpha}{x_n} + \frac{\beta}{x_n^2} \tag{4}$$

which preserves the two-form $(dx_{n-1} \wedge dx_n)/(x_{n-1}x_n)$ and has the conserved quantity

$$J = x_{n-1}x_n + \alpha \left(\frac{1}{x_{n-1}} + \frac{1}{x_n} \right) + \frac{\beta}{x_{n-1}x_n}. \tag{5}$$

A symplectic map of the plane with a conserved quantity is the discrete analogue of a

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Hamiltonian system with one degree of freedom, and hence is integrable in the sense of Liouville (see chapter 10 in [20]). Thus the map defined by (4) is integrable and belongs to the well studied class [2, 6] of rational mappings of the plane that preserve an algebraic curve, in this case the curve of genus one defined by (5). In fact, certain integer sequences satisfying the Somos-4 recurrence (3) were already known to number theorists by the name of elliptic divisibility sequences [21], and these continue to be the subject of active research due to the way that new prime divisors appear therein.

The main result proved below is the following

Theorem *Given a polynomial $f(x)$ of degree d having the form*

$$f(x) = x^M F(x)$$

for a non-constant polynomial F with $F(0) \neq 0$, the recurrence

$$x_{n+1} x_{n-1} = f(x_n) \tag{6}$$

possesses the Laurent property if and only if one of the following three cases holds:
(i) $M = 0$ and, for all x, the polynomial f satisfies

$$f(x) = \pm x^d f(1/x), \quad \text{with } f(0) = 1 \text{ for } d \neq 2; \tag{7}$$

(ii) $M = 1$ and, for all x, the polynomial F (of degree D) satisfies

$$F(x) = \pm x^D F(1/x), \quad \text{with } F(0) = 1 \text{ or } -1 \text{ for } D \neq 1; \tag{8}$$

(iii) $M \geq 2$ and F is an arbitrary non-constant polynomial.

Moreover, up to the freedom to rescale x_n, in cases (i) and (ii) the respective conditions (7) and (8) are also necessary and sufficient for the singularities of (6) to

be confined immediately, while in case (iii) the singularity confinement test is failed.

Clearly to obtain Laurent polynomials it is necessary for $f(x)$ to be a polynomial in x, and the recurrence (6) is generated by iterating the rational mapping of the plane defined by

$$\phi: \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ f(y)/x \end{pmatrix}, \tag{9}$$

which preserves the two-form

$$\omega = (xy)^{-1}\, dx \wedge dy. \tag{10}$$

For (6) to have the Laurent property it is required that $x_n \in \mathcal{R} := \mathbb{Z}[\mathbf{c}, x_0^{\pm 1}, x_1^{\pm 1}]$ for all n, where $\mathbf{c}$ denotes the coefficients of f. To begin with, we take the generic case $f(0) = \lambda \neq 0$. Given the two initial data x_0, x_1, the next iterates are $x_2 = f(x_1)/x_0$ and $x_3 = f(x_2)/x_1$, which are both in $\mathcal{R}$ for any f. The first place where a division must occur is for $x_4 = f(x_3)/x_2$: if $x_4 \in \mathcal{R}$ then $f(x_1) = x_2\, x_0$ must divide $f(x_3)$. Now every Laurent polynomial can be written as a polynomial divided by a monomial $x_0^\ell x_1^m$, and the ring $\mathcal{R}$ has the structure of a unique factorization domain, with units given by the monomials $\pm x_0^\ell x_1^m$ for integers ℓ, m. Thus one can do modular arithmetic with the elements of $\mathcal{R}$ in the usual way, and note that x_1 and x_2 are coprime. Then $x_3 \equiv f(0)/x_1 \equiv \lambda/x_1 \bmod x_2$ and hence $f(x_3) \equiv f(\lambda/x_1) \bmod x_2$, so for divisibility of $f(x_3)$ by $f(x_1)$ we must have that, for arbitrary x_1,

$$\lambda f(x_1) = \mu\, x_1^d\, f(\lambda/x_1) \tag{11}$$

where $d = \deg f$ and the constant μ is the leading coefficient of f. Comparing the constant term on each side of (11) gives $\lambda^2 = \mu^2 \lambda^d$, so $\mu = \pm \lambda^{1-d/2}$. If $d \neq 2$ then to have polynomials in λ forces the choice $\lambda = 1$ (and in any case, the freedom in λ can be removed by rescaling x_n), so we find $\mu = \pm 1$ for all d, and thus the necessary

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condition for the recurrence (6) to have the Laurent property is that the polynomial f satisfies the reciprocal property (7).

The condition (7) also turns out to be sufficient to ensure that $x_n \in \mathcal{R}$ for all n, and also implies that (6) passes the singularity confinement test. To see that this is sufficient for the Laurent property, one can apply Fomin and Zelevinsky's Caterpillar Lemma [14], but a direct proof is given here for completeness. Take as the inductive hypothesis that $x_j \in \mathcal{R}$ for $0 \leq j \leq n$ with all adjacent pairs x_j, x_{j+1} being coprime in this range. Then $x_n = f(x_{n-1})/x_{n-2} \equiv f(0)/x_{n-2} \equiv 1/x_{n-2} \bmod x_{n-1}$, so $x_{n+1} x_{n-1} = f(x_n)$ and then using (7) and (6) once more gives $f(x_n) \equiv f(1/x_{n-2}) \equiv \pm x_{n-2}^{-d} f(x_{n-2}) \equiv \pm x_{n-2}^{-d} x_{n-3} x_{n-1} \equiv 0 \bmod x_{n-1}$, whence $x_{n-1}|f(x_n)$ and $x_{n+1} \in \mathcal{R}$ as required. Furthermore, suppose that p is an irreducible element of $\mathcal{R}$ such that $p|x_{n+1}$; then $p|f(x_n)$ and $f(x_n) \equiv 1 \bmod x_n$, so $p \nmid x_n$ and hence x_n and x_{n+1} are coprime, which completes the inductive step. This proves that $x_n \in \mathcal{R}$ for all positive indices n, and the result extends to negative n by the reversibility of (6).

As for singularity confinement, note that a singularity can only occur in (6) if one of the iterates becomes zero. So let $x_{n-3} = a$ and $x_{n-2} = r + \epsilon$ where r is any one of the (generically distinct) roots of f. Thus $x_{n-1} = a^{-1} f'(r) \epsilon + O(\epsilon^2) \to 0$ as $\epsilon \to 0$, while $x_n = f(0)/r + O(\epsilon) = 1/r + O(\epsilon)$, which gives $x_{n+1} = (f'(r))^{-1} a \epsilon^{-1} f(1/r + O(\epsilon)) = O(1)$ because $f(r) = 0$ implies $f(1/r) = 0$ by (7), and thereafter all the terms are finite (and non-zero) as $\epsilon \to 0$. So we see that the singularity is confined at the first possible stage for any mapping of the form (6) with $f(0) \neq 0$ that has the Laurent property. Conversely, it is easy to see that if we start from (6) with a polynomial f such that $f(0) \neq 0$, we can always scale so that $f(0) = 1$, and if we require that a zero is confined immediately, then for any root r of f, the reciprocal value $1/r$ must also be a root, which implies that (7) must hold, and hence the mapping has the Laurent property.

In the above considerations we imposed the restriction $f(0) \neq 0$. We now describe

the situation for the case $f(0) = 0$, and so set $f(x) = x^M F(x)$ with integer $M \geq 1$, $F(0) \neq 0$ and $\deg F = D \geq 1$. There are two further cases to consider, according to whether $M = 1$ or $M \geq 2$. In case (ii), we find that for the Laurent property to hold we must have

$$f(x) = x\, F(x), \qquad F(x) = \pm x^D F(1/x) \tag{12}$$

for all x, with $F(0) = 1$ or -1 for $D \neq 1$. So F must satisfy the same reciprocal property as for f in the generic case, and once again this condition is also sufficient; the proof of the Laurent property in this case is slightly more involved and will be presented elsewhere. Similarly to the argument for (7), it is easy to verify that the condition (12) also implies that the mapping (6) passes the singularity confinement test. The other case (iii) is somewhat different, for upon taking

$$f(x) = x^M F(x), \qquad M \geq 2, \qquad F \text{ arbitrary}, \tag{13}$$

we find that the mapping defined by (6) gives $x_n \in \mathcal{R}$ for all n. To see this, take as the inductive hypothesis that x_{n-1}, x_n and $\rho_n = (x_n/x_{n-1}) \in \mathcal{R}$, and then write $\rho_{n+1} = (x_{n+1}/x_n) = \rho_n\, x_n^{M-2} f(x_n) \in \mathcal{R}$ by the hypothesis, so $x_{n+1} = \rho_{n+1}\, x_n \in \mathcal{R}$ as required. However, to see if a zero is confined in such a mapping we set $x_n = a$, $x_{n+1} = r + \epsilon$ where $F(r) = 0$, so that $x_{n+2} = C_2\, \epsilon + O(\epsilon^2) \to 0$ as $\epsilon \to 0$, where $C_2 = a^{-1} r^M F'(r)$. Subsequent terms have $x_{n+3} \sim C_3\, \epsilon^M$, $x_{n+4} \sim C_4\, \epsilon^{M^2-1}$, $x_{n+5} \sim C_5\, \epsilon^{M^3-2M}$ etc. for certain (non-zero) constants C_j, so the powers of ϵ continue to grow and the zero is *not* confined. (However, note that we have explicitly excluded the trivial case $F = \beta = \text{constant}$, $f(x) = \beta x^M$, which always has the Laurent property and satisfies singularity confinement in the sense that $x = 0$ cannot be reached from any non-zero initial data.)

We have seen that there is a close connection between the Laurent property and singularity confinement for discrete equations of the form (6), but what about the

integrability of such mappings? For a measure-preserving mapping of the plane to be integrable it must have a conserved quantity. We can start by considering the lowest degree examples of polynomials f. If $d = 1$ then

$$x_{n+1}\,x_{n-1} = \gamma\,x_n + \delta, \tag{14}$$

and any such map is integrable because it has a conserved quantity K defining an elliptic curve, i.e. K equals

$$x_{n-1} + x_n + \frac{\gamma(x_{n-1}^2 + x_n^2) + (\delta + \gamma^2)(x_{n-1} + x_n) + \gamma\delta}{x_{n-1}x_n}. \tag{15}$$

Equation (14) is bilinearized via $x_n = \tau_{n+3}\tau_{n-2}/(\tau_{n+1}\tau_n)$, yielding the special Somos-7 recurrence

$$\tau_{n+4}\,\tau_{n-3} = \gamma\,\tau_{n+3}\,\tau_{n-2} + \delta\,\tau_{n+1}\,\tau_n,$$

which can be related to a Somos-5 recurrence and thence solved in terms of elliptic sigma functions [17]. However, if we require that (14) itself should possess the Laurent property, then either $\delta = 0$ with γ arbitrary and the map cycles with period 6, or we are in the situation (7) so that $\gamma = \pm 1$ with $\delta = 1$ and the map cycles with period 5. It is interesting to note that in the latter case (fixing $\gamma = 1$) this map is equivalent to the functional relation that appears in the thermodynamic Bethe ansatz for an A_2 scattering theory [22].

The case $d = 2$ is special: if $\lambda = f(0) \neq 0$ then the Laurent property requires (7) to hold, and there is the extra freedom to leave $\lambda \neq 0$ arbitrary. Then there are two choices of sign, giving either

$$x_{n+1}\,x_{n-1} = x_n^2 + \nu\,x_n + \lambda, \tag{16}$$

or alternatively $x_{n+1}\,x_{n-1} = -x_n^2 + \lambda$. The second choice can be transformed into the

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first by taking $x_n = \kappa_n \tilde{x}_n$ where $\kappa_n^2 = 1$ and $\kappa_{n+1}\kappa_{n-1} = -1$ for all n, so we can just consider (16) which has a conserved quantity that defines a conic, namely

$$L = \frac{x_{n-1}}{x_n} + \frac{x_n}{x_{n-1}} + \nu \left(\frac{1}{x_{n-1}} + \frac{1}{x_n} \right) + \frac{\lambda}{x_{n-1}x_n}. \tag{17}$$

Furthermore, the iterates also satisfy a linear recurrence of the form (1), viz.

$$x_{n+1} + x_{n-1} = Lx_n - \nu,$$

so in terms of the initial data we have $L = L(x_0, x_1) \in \mathcal{R}$ and also $x_n \in \mathbb{Z}[\nu, x_0, x_1, L] \subset \mathcal{R}$ for all n, which is even stronger than the Laurent property. When $d = 2$ and $f(0) = 0$ there are further sub-cases. If (12) holds then (up to rescaling $x_n \to \nu^{-1}x_n$) either $f(x) = x(x+1)$, which is just a special case of (16), or we have the opposite choice of sign and the mapping is given by

$$x_{n+1}\,x_{n-1} = -x_n(x_n + 1), \tag{18}$$

which is also integrable, having the conserved quantity

$$\tilde{J} = \frac{x_{n-1}^4 + x_n^4 + (x_{n-1} - x_n)^2(2x_{n-1} + 2x_n + 1)}{x_{n-1}^2 x_n^2}.$$

Moreover, for (18) the iterates also satisfy the sixth order linear recurrence

$$x_{n+6} + (\tilde{J} - 1)(x_{n+4} - x_{n+2}) - x_n = 0,$$

which provides an alternative proof that $x_n \in \mathcal{R}$ based on the the fact that $\tilde{J} \in \mathcal{R}$ and $x_j \in \mathcal{R}$ for $j = 0, 1, \ldots, 5$.

For $d \geq 3$ all of the maps (6) with the Laurent property are non-integrable. To see this, one can count the growth of degrees to show that the algebraic entropy is non-

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zero [7], but a simpler test to apply is Halburd's Diophantine integrability criterion [13]. For a rational number $x = p/q$ the logarithmic height is $h(x) = \log \max\{|p|, |q|\}$, and Halburd's test requires that for rational-valued maps to be integrable, $h(x_n)$ must grow no faster than a polynomial in n. Suppose that a sequence of (real or complex) iterates is such that $|x_n| \to \infty$ and $\Lambda_n = \log|x_n| \sim C\zeta^n$ for real $\zeta > 1$ and some $C > 0$. Then taking logarithms of both sides of (6) gives $\Lambda_{n+1} + \Lambda_{n-1} - d\Lambda_n \approx 0$ and hence with $d \geq 3$ we find $\zeta = (d + \sqrt{d^2 - 4})/2 > 1$ as required. Now for equations (6) having the Laurent property, if we take f to have integer coefficients and set $x_0 = x_1 = 1$ then all x_n are integers, so that $\Lambda_n = h(x_n)$. Moreover, if all the coefficients of f are positive then the terms of the integer sequence will have precisely these asymptotics, so that the logarithmic height grows exponentially and $\lim_{n\to\infty}(\log h(x_n))/n = \log \zeta$ (which also happens to be the value of the algebraic entropy for these maps).

To understand the deep connection between singularity confinement and the Laurent property, we propose to extend the above results in at least three directions. Firstly, given a pair of polynomials f_1, f_2 each satisfying (7), it is simple to prove that the composition $\phi_1 \cdot \phi_2$ of the corresponding pair of maps also has the Laurent property; the choice $f_1(x) = x^b + 1$, $f_2(x) = x^c + 1$ generates a cluster algebra of rank 2 [23]. Secondly, there are many higher order discrete equations (integrable and non-integrable) with the Laurent property. Thirdly, this property can also apply to non-autonomous equations, such as the bilinear forms of discrete Painlevé equations.

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